Stable ∞-categories.

In doing derived algebraic geometry most of the ∞-categories we will encounter will be ∞-categorical analogues of abelian (or additive) categories. The theory of these special ∞-cats is very rich and easier to work with than arbitrary ∞-categories.

We need a couple of defitions:

- An ∞-category $\mathcal{C}$ is said to be *pointed* if it has an object $0 \in \mathcal{C}$ which is both initial & final;
- A push out diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \quad (\Delta)$$

is called a *cofiber sequence*. (Sometimes abbreviated to: $X \to Y \to Z$, but notice the map $X \xrightarrow{0} Z$ is some data, together w/ the witness of commutativity of (Δ)).

- A pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is called a *fiber sequence*. $(X \to Y \to Z)$.

- Given a morphism $X \xrightarrow{f} Y$ a

a *cofiber of f* is a push out diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{Cofib}(f) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

a *fiber of f* is a pullback diagram:

$$\begin{tikzcd} \mathrm{Fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

.

<u>Def'n:</u> An ∞-category $\mathcal{C}$ is said to be <u>stable</u> if:

(a) $\mathcal{C}$ is pointed;

(b) every morphism in $\mathcal{C}$ has a fiber & cofiber;

(c) a diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$
is a fiber sequence iff it is a cofiber sequence.

Here is a more succint repackaging of the above, and also consequence of these conditions for finite limits & colimits.

<u>Prop:</u> A pointed category $\mathcal{C}$ is stable iff:

(i) $\mathcal{C}$ admits all finite limits & colimits; <u>and</u>

(ii) every pushout square is a pullback square.

The following result gives an idea of what the underlying homotopy category of an ∞-category ~~[illegible]~~ can look like:

<u>Prop:</u> Given $\mathcal{C}$ a stable ∞-category, $h\mathcal{C}$ is a triangulated category.

<u>Idea of proof:</u> $X \to Y \to Z$ in $h\mathcal{C}$ is a distinguished triangle iff it is the image of a (co)fiber sequence in $\mathcal{C}$. In this case, we get a map:
$$Z \to X[1] := 0 \coprod_X 0$$
from
$$0 \coprod_X Y \to 0 \coprod_X 0.$$

Then check the axioms of a triangulated category. [HA, Thm 1.1.2.14].

⚠ One gets the octahedral axiom for free.

Warning: The analogue of this Lemma for triangulated categories does not work.

E.g.: in $D(k)$

$$\begin{tikzcd} k \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & k[-1] \arrow[d, "\alpha"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & k[-1] \arrow[r] & k[-1] \end{tikzcd}$$

$\left(\begin{array}{l}\alpha = 0 \\ \text{or } = id_{k[-1]}.\end{array}\right)$.

One of the advantages of working w/ a stable ∞-category is that we have fiber and cofiber functors.

Lemma: For $\mathcal{C}$ a stable ∞-category, one has a functor:

$$Cfib : Fun([1], \mathcal{C}) \to \mathcal{C}.$$
$$(X \to Y) \mapsto Z \quad \text{for some} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}.$$

Pf: Let $L_0^2 := \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2 & \end{tikzcd}$ and $i : [1] \to L_0^2$ the evident inclusion.

We claim that $RKE_i(f) : L_0^2 \to \mathcal{C}$, $f \in Fun([1], \mathcal{C})$ exists. and $RKE_i(f)(2) \simeq 0$.

Indeed, by Lemma (b) of last we need to check:

$$\lim_{[1] \times_{[1]} L_0^{2\,0/}} f = \lim_{[1]} f = ev_0 \circ f, \quad \lim_{[1] \times_{[1]} L_0^{2\,1/}} f \simeq \lim_{\{1\}} f = ev_1 \circ f$$

and $\lim_{[1] \times_{[1]} (L_0^2)^{2/}} f = \lim_{\emptyset} f \simeq *_{\mathcal{C}} = 0_{\mathcal{C}}$.

Similarly, let $j : L_0^2 \hookrightarrow \begin{tikzcd} \overset{0}{\bullet} \arrow[r] \arrow[d] & \overset{1}{\bullet} \arrow[d] \\ 2 \arrow[r] & 3 \end{tikzcd} = [1] \times [1]$

We claim lemma (a) from last time gives: $\underset{\overline{f}}{\underset{\parallel}{LKE_j(RKE_i(f))}} : [1] \times [1] \to \mathcal{C}$.

moreover $ev_3 \circ \overline{f} \simeq 0 \underset{ev_0 \circ f}{\amalg} ev_1 \circ f$.

So $Cfib(f) := ev_3 \circ \overline{f}$. $\blacksquare$

<u>Rk:</u> One can also define functors $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ whose value on objects is given by:

$$\Sigma_{\mathcal{C}}(X) = 0 \coprod_X 0, \qquad \Omega_{\mathcal{C}}(X) = 0 \times_X 0.$$

For instance, let $\mathcal{M}^{\Sigma} \subseteq \mathrm{Fun}([1]\times[1], \mathcal{C})$

$$= \left\{ \begin{array}{ccc} X & \to & 0 \\ \downarrow & & \downarrow \\ 0' & \to & Y \end{array} \right\}.$$

Then lemma from last time is equivalent to $\mathcal{M}^{\Sigma} \xrightarrow{ev_0} \mathcal{C}$, $(-) \mapsto X$ is a trivial fibration.

Let $s : \mathcal{C} \to \mathcal{M}^{\Sigma}$ be a section, we let

$$\Sigma_{\mathcal{C}} := ev_3 \circ s : \mathcal{C} \to \mathcal{C}. \qquad ev_3 : \mathcal{M}^{\Sigma} \to \mathcal{C}, \ (-) \mapsto Y.$$

One also denote these functors by: $\Omega_{\mathcal{C}}(X) := X[-1]$ and $\Sigma_{\mathcal{C}}(X) := X[1]$.

The following is a useful result to check if some ∞-category is stable

[HA, Prop. 1.4.2.11] & [HA, Cr. 1.4.2.27]. <u>Prop:</u> Let $\mathcal{C}$ be a pointed ∞-category, TFAE:

(i) $\mathcal{C}$ is stable;

(ii) $\mathcal{C}$ has finite limits and $\Omega_{\mathcal{C}}$ is an equivalence;

(iii) $\mathcal{C}$ has finite colimits and $\Sigma_{\mathcal{C}}$ is an equivalence.

Let's finally discuss some examples of stable categories.

Example: (i)

Let Spc_* denote the ∞-category of pointed spaces, notice $* \simeq 0_{Spc_*}$

So $\Omega : Spc_* \to Spc_*$ is the pointed loop functor.
$$X \mapsto * \times_X *$$

It is well-known that Ω is not an equivalence. (i.e. classical works on characterizing ~~iter~~ loop spaces.). In fact, objects in the essential image has some form of group structure.
Also Ω is not fully faithful. (not full.).

Defn: The ∞-category of spectra is defined as the following limit (in Cat_∞):
$$Spctr := \lim\left(\cdots \xrightarrow{\Omega} Spc_* \xrightarrow{\Omega} Spc_*\right).$$

Spctr is stable. Clear from the last Proposition.

(ii) Let k be a commutative ring and consider $\mathcal{D}(k)$ its derived ∞-category

Claim: $\mathcal{D}(k)$ is stable.
① check $Ch(Mod_k^{inj})$ has finite homotopy colimits.
② check $\Sigma_{\mathcal{D}(k)}$ is an equivalence.

In $Ch(Mod_k^{inj.})$ a pushout diagram
$$\begin{array}{ccc} M^* & \xrightarrow{f} & (M')^* \\ \downarrow & & \downarrow \\ N^* & \to & (N')^* \end{array}$$

where f is degree wise split is a also a homotopy pushout diagram.

Thus, ~~we can calculate~~ the ordinary pushout:
$$\begin{array}{ccc} M^\cdot & \to & M^\cdot \oplus M^{\cdot-1} \\ \downarrow & & \downarrow \\ 0 & \to & \Sigma_{\mathcal{D}(k)}(M^\cdot) \end{array}$$
computes the ~~usual~~ pushout in $\mathcal{D}(k)$.
Since $\Rightarrow \Sigma(M^\cdot) \cong M^{\cdot-1}$ which is an equiv.

Def'n:

A functor $F: \mathcal{C} \to \mathcal{D}$ between stable ∞-cats is exact if

- it preserves pullbacks, equivalently pushouts.

We want to introduce a nice class of stable ∞-cats.

Def'n: An ∞-category $\mathcal{C}$ is said to be cocomplete if it contains all colimits.

For $\mathcal{C}$ stable TFAE:

(i) $\mathcal{C}$ is cocomplete;

(ii) $\mathcal{C}$ admits all filtered colimits;

(iii) $\mathcal{C}$ admits all direct sums. (coproducts).

A functor $F: \mathcal{C} \to \mathcal{D}$ between cocomplete stable ∞-categories is continuous if F preserves colimits, i.e. F is exact & preserve filtered colimits.

We will be interested in the ∞-category: $Cat_\infty^{st.}$ where objects are cocomplete ∞-categories & functors are continuous fctors.

Universal properties of Spc & Spctr.

Lemma: Let $z: * \hookrightarrow Spc$ be the inclusion of trivial ∞-cat. into spaces; $\forall \mathcal{C} \in Cat_\infty$ one has:

$$Fun(*, \mathcal{C}) \underset{(-)\circ z}{\overset{LKE_z}{\rightleftarrows}} Fun(Spc, \mathcal{C}) \quad \text{s.t.}$$

$$Fun(*, \mathcal{C}) \xrightarrow[LKE_z]{\simeq} \underset{\text{colimit preserving fctors.}}{Fun^L(Spc, \mathcal{C})} \subseteq Fun(Spc, \mathcal{C}).$$